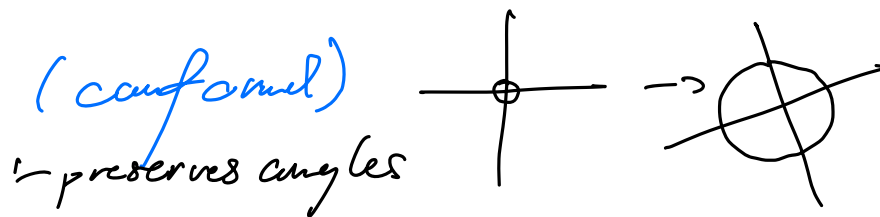


Review of complex analysis + Riemann surfaces:

Rotation + dilation: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = (a^2 + b^2) \cdot \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$,
 $z = a + bi$



$U \subset \mathbb{C} = \mathbb{R}^2$, $f: U \rightarrow \mathbb{C}$ is holomorphic if
 $f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$ exists. $f = u + iv$
 $z = x + iy$ (CR eq's)
 implies that the derivative exists $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$Df: \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$ is complex valued: $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thm. $f: U \rightarrow \mathbb{C}$ is hol. iff f is analytic:

$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a neighborhood of z_0

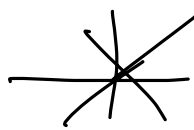
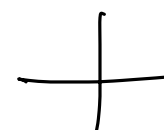
Thm. If $f_n: U \rightarrow \mathbb{C}$ conv. unif. to f , then f is hol. and $f_n' \rightarrow f'$ loc. unif.

Def. $f: U \rightarrow \mathbb{C}$ is meromorphic if for each $z_0 \in U$,
 $\exists n \geq 0$ s.t. $(z - z_0)^n f(z)$ is hol.

Cor. (Degree of vanishing) f is mer. $\exists! n \in \mathbb{Z}$

$f(z) = (z - z_0)^n \underbrace{h(z)}_{\text{hol.}}$, $h(z) \neq 0$ near z_0 .

$z \mapsto z^n$

 $\rightarrow$  n times

Def. A Riemann surface X is a second-countable Hausdorff topological space so that any $z \in X$ has a pair (U, ϕ_U) called a chart, where

- 1) $z \in U$ 2) $\phi_U: U \rightarrow \mathbb{C}$ is homeo onto its image

3) If $(U, \phi_U), (V, \phi_V)$ are charts, $U \cap V \neq \emptyset$, $\phi_U \circ \phi_V^{-1}$ is holomorphic. (Transition function)

Def. X, Y are R.S. $f: X \rightarrow Y$ is holo. if $\phi_V \circ f \circ \phi_U^{-1}$ is holo for any pair of charts

(U, ϕ_U) of X , (V, ϕ_V) of Y . Here $\mathbb{C} \leftrightarrow \hat{\mathbb{C}}$
 $(\text{holo } \mathbb{C} \rightarrow \hat{\mathbb{C}})$

f is isomorphism if f, f^{-1} are holo

Ex: 1) $\mathbb{C}$, 2) $U \subset \mathbb{C}$ open, 3) $\mathbb{D} \cong \mathbb{H}$ $z \mapsto i \frac{(1-z)}{(1+z)}$

$$\begin{array}{c} \text{Diagram: A circle with points } i, -i, 1, -1 \text{ on the boundary. Below it, a mapping } \frac{i-w}{i+w} \leftarrow w \end{array}$$

4) $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. $\mathbb{C} \times \{0, 1\} / (z, 0) \sim (\frac{1}{z}, 1)$
 $\infty = (0, 1)$

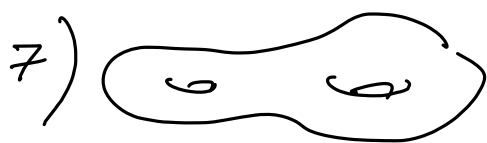
5) Λ lattice (discrete subgroup of $\mathbb{C}$) $\sim \mathbb{C}/\Lambda$

$\mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda$ Λ is discrete

$\sim \in R.S. : \pi : \mathbb{C} \rightarrow \mathbb{C}/\Lambda$ - down to a torus,
 $\exists U \subset \mathbb{C}, U \ni 0$ s.t. $\pi|_U$ is homeo onto its image.

Let $(\pi(U), \pi|_U^{-1})$ be a chart around $[0] \in \mathbb{C}/\Lambda$.

For $[z] \in \mathbb{C}/\Lambda$, translate the chart.



Lemma (Max. Mod. Principle) If $f: U \rightarrow \mathbb{C}$ is holomorphic, U open + connected, $z_0 \in U$, $|f(z_0)| \geq |f(z)| \forall z$, then $f(z) = f(z_0) \forall z$.

LaScheur & Lemma If $f: \mathbb{D} \rightarrow \mathbb{D}$ holomorphic, $f(0) = 0$, then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$. If $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$, then $f(z) = \lambda z$ for $|\lambda| = 1$.

Proof. $g(z) = \begin{cases} f(z)/z & z \neq 0 \\ f'(0) & z = 0 \end{cases}$ is holomorphic.

For each r , by MRP, $\sup_{|z| \leq r} |g(z)| \leq \sup_{|z|=r} |g(z)| \leq \frac{1}{r}$.

Take $r \rightarrow 1^-$, $|f(z)| \leq |z|$, $|f'(0)| = 1$.

if $|g(z_0)| = 1$, then g is constant by UoP.

Prop. All holomorphic $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ are rational functions.

Proof. Subtract the fin. many zeroes and poles from f to get a bounded function on $\mathbb{C}$.

Cor. $\text{Aut}(\hat{\mathbb{C}}) = \text{PSL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det = 1 \right\} / \pm I$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

Möbius transformations

Classification of Möbius: $M \in \text{PSL}(2, \mathbb{C})$

Jordan form: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$
"conj. in $\text{PSL}(2, \mathbb{C})$ by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ "

1) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{tr}^2 = 4$
 $z \mapsto z + 1$ rep in $\text{SL}(2, \mathbb{C})$

2) $0 \leq \text{tr}^2 < 4$

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

3) $\text{tr}^2 > 4$

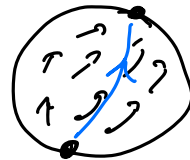
$$\begin{pmatrix} e^\theta & 0 \\ 0 & e^{-\theta} \end{pmatrix}$$



"parabolic"

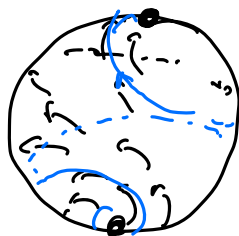


"elliptic"



"hyperbolic"

4) $\text{tr}^2 \notin [0, 4]$



"loxodromic"

• Möbius carry circle to circle

Cor. $\text{Aut}(\mathbb{C}) = \{z \mapsto az + b, a \neq 0\}$

since $\text{Aut}'s$ are proper

Proof. $\text{Aut}(\mathbb{C}) \subset \text{Aut}(\mathbb{D})$ *following Blaschke factor*

Prop. $\text{Aut}(\mathbb{D}) = \left\{ z \mapsto e^{i\theta} \frac{z-a}{1-\bar{a}z} \mid a \in \mathbb{D} \right\}$

Proof. $g \in \text{Aut}(\mathbb{D}), g(a) = 0$. Then

$$f_a(z) = \frac{z-a}{1-\bar{a}z}, \quad f_a(a) = 0, \quad h = g \circ f_a^{-1}, \quad h(0) = 0.$$

By Schwarz, $|h'(0)| \leq 1, \quad |(h^{-1})'(0)| \leq 1, \quad \text{so}$
 $= |h'(0)|^{-1}$

$$|h'(0)| = 1, \quad g(z) = e^{i\theta} f_a(z)$$

Cor. $\text{Aut}(\mathbb{H}) = \text{PSL}(2, \mathbb{R}) = \{ A \in \text{PSL}(2, \mathbb{C}) \mid A(\mathbb{H}) = \mathbb{H} \}$

Classification of $\text{PSL}(2, \mathbb{R})$: Extends to $\bar{\mathbb{H}} = \bar{\mathbb{D}}$

Real Jordan: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$

1) Parabolic $\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$

$\rightarrow \rightarrow \rightarrow$
 $\leftarrow \leftarrow \leftarrow$



$tr^2 = 4$
1 Fixed pt on ∂

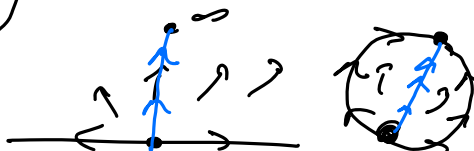
2) Elliptic $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ *rotating*



1 fixed pt in interior
 $0 \leq tr^2 < 4$

3) Hyperbolic $tr^2 > 4$

$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$



2 fixed pt on ∂